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INTERNAL WAVES IN THE MIDDLE LAYER OF A THREE-LAYER ATMOSPHERE FROM SOURCES LOCATED IN THE LOWER LAYER†

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Oscillations in the middle layer of a three-layer atmosphere induced by sources located in the lower layer are investigated. In every layer the Brunt–Väisälä frequencies are constants and increase with height. The general solution obtained by the methods of integral transforms is presented in the form of series. It is shown that, up to terms $O(t^{-1})$, only a finite number of terms of the series make a fundamental contribution to the asymptotic form as $t \to \infty$. The asymptotic form is investigated by standard methods of the theory of asymptotic estimates of integrals. © 2004 Elsevier Ltd. All rights reserved.

1. STATEMENT AND FORMAL SOLUTION OF THE PROBLEM

Consider an ideal atmosphere that fills three-dimensional space and is split into three layers, with Brunt–Väisälä (BV) frequencies that are constant in each layer. We will choose the length and time scale in such a way that the thickness of the middle layer and BV frequency is equal to unity. It is assumed that $N_1 < N_2 < N_3$, where N_1 is the BV frequency in the lower layer, $N_2 = 1$, and N_3 is that in the top layer. Previously [1] the less realistic case when $N_1 = N_3 \neq 1$ was considered. (It is well known that in the actual atmosphere the BV frequency increases with height.) The origin of a Cartesian system of coordinates is chosen on the lower boundary of the middle layer. Point or distributed sources are located in the lower layer.

In the linear formulation the solution of the problem of internal waves produced by these sources is expressed in terms of various convolutions over spatial variables with a continuous and bounded solution of the equation with piecewise-constant coefficients.

$$\frac{\partial^2(\Delta w)}{\partial t^2} + N_k^2 \Delta_2 w = \frac{1}{4\pi} Q'(t)\delta(x)\delta(y)\delta(z+c), \quad k = 1, 2, 3 \tag{1.1}$$

The function $Q(t)$ is continuously differentiable and finite with a support at $[0, T]$, $c > 0$.

The solution of problem (1.1) was obtained in [1] by the method of integral transforms. In the middle layer the disturbances are given by the formula

$$w = \frac{1}{16\pi^3 i} \int_0^T Q(\tau) \int_0^{+\infty} \int_{C-i\infty}^{C+i\infty} \varphi(u, p, z) J_0(ru) e^{p(t-\tau)} du dp d\tau, \quad 0 < z < 1 \tag{1.2}$$

where

$$\varphi(u, p, z) = -e^{-u\gamma c/p} \frac{(\beta+\omega)e^{u\omega(1-z)/p} - (\beta-\omega)e^{-u\omega(1-z)/p}}{(\beta+\omega)(\omega+\gamma)e^{u\omega/p} + (\beta-\omega)(\omega-\gamma)e^{-u\omega/p}} \tag{1.3}$$

$$\omega = \sqrt{1+p^2}, \quad \beta = \sqrt{N_3^2+p^2}, \quad \gamma = \sqrt{N_1^2+p^2} \tag{1.4}$$

It is well known that the middle layer acts as a waveguide. Oscillations produced by a source can disrupt the functions of the waveguide.

In a similar way we can write formulae for disturbance in the upper and lower layers.

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2. TRANSFORMATION OF THE SOLUTION

Arguments similar to those above [1] show that when $u > 0$, Re$p \neq 0$ the denominator of the function φ cannot vanish. If we use the fact that $N_1 < 1 < N_3$, it turns out that when Re$p = 0$ the denominator of the function φ can only vanish at the points $p = \pm i$, but at these points the numerator also vanishes. The function φ has finite limits as $x \to \pm i$ and can be assumed to be continuous at these points. By virtue of Cauchy's theorem one can change to integration over the imaginary axis in formula (1.2). If the function φ is represented in the form of a difference, the integral can be represented in the form of a difference of two integrals, taken in the sense of the principal value. At complex-conjugate points of the imaginary axis the integrand in formula (1.2) takes complex-conjugate values. If when $t > T$ we replace the function $f(p, t)$ in formula (1.2) by $f(-p, t)$, its right-hand side vanishes by virtue of Cauchy's theorem. All this enables us to write formula (1.2) in the form

$$w = \frac{1}{4\pi^3}\mathrm{Im}\int_0^T Q(\tau)\int_0^{+\infty+i\infty}\int_0 \varphi(u, p, z)J_0(ru)\mathrm{ch}\,p(t-\tau)\,du\,dp\,d\tau \tag{2.1}$$

Using formulae (1.3) and (1.4) we obtain

$$\varphi(u, p, z) = \sum_{n=0}^{\infty}(-1)^{n+1}C(p)A^n(p)B^n(p)\exp\left(-\frac{u}{p}((z+2n)\omega+\gamma c)\right) +$$

$$+\sum_{n=0}^{\infty}(-1)^n C(p)A^n(p)B^{n+1}(p)\exp\left(-\frac{u}{p}((2-z+2n)\omega+\gamma c)\right) \tag{2.2}$$

$$C(p) = \frac{1}{\omega+\gamma},\quad A(p) = \frac{\omega-\gamma}{\omega+\gamma},\quad B(p) = \frac{\beta-\omega}{\beta+\omega} \tag{2.3}$$

Substituting expression (2.2) into formula (2.1), using formulae (1.4) and evaluating the integrals over the variable u, we obtain

$$\begin{aligned} w &= w_1 - w_2 \\ w_1 &= \frac{1}{4\pi^3}\int_0^T Q(\tau)\Phi_1(r, z, t-\tau)\,d\tau \\ w_2 &= \frac{1}{4\pi^3}\int_0^T Q(\tau)\Phi_2(r, 2-z, t-\tau)\,d\tau \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Phi_1(r, z, t) &= \sum_{n=0}^{\infty}(-1)^{n+1}\mathrm{Re}\int_0^{\infty}\frac{\Gamma_n(q)\cos(qt)\,dq}{\sqrt{f(q, r, z+2n)}} \\ \Gamma_n(q) &= qC(iq)A^n(iq)B^n(iq) \\ f(q, r, z) &= r^2q^2 - g^2(q, z),\quad g(q, z) = z\sqrt{1-q^2} + c\sqrt{N_1^2-q^2} \end{aligned} \tag{2.5}$$

The expression for the function $\Phi_2(r, z, t)$ is obtained by substituting the coordinate $2-z$ for z in formula (2.5) and multiplying the integrand by $B(iq)$.

The function $f(q, r, z+2n)$ vanishes only when

$$rq = g(q, z+2n),\quad 0 < q < N_1 \tag{2.6}$$

To solve Eq. (2.6), we put

$$N_1\sin\varphi = \sqrt{\frac{N_1^2-q^2}{1-q^2}},\quad 0 \le \varphi \le \frac{\pi}{2},\quad N = \frac{N_1}{\sqrt{1-N_1^2}} \tag{2.7}$$

Expressing q from this formula and substituting the result into Eq. (2.6), we obtain

$$q(\varphi) = \frac{N\cos\varphi}{\sqrt{1+N^2\cos^2\varphi}}, \quad rN\cos\varphi - c\sin\varphi = z+2n \tag{2.8}$$

Assuming

$$R = \sqrt{r^2N^2+c^2}, \quad rN = R\sin\alpha, \quad c = R\cos\alpha$$

we write Eq. (2.8) in the form

$$\sin(\alpha-\varphi) = (z+2n)/R \tag{2.9}$$

Equation (2.9) can have a solution only when $z + 2n/R \leqslant 1$. Assuming this condition is satisfied, we put

$$\sin(\beta(z, r)) = z/R, \quad 0 \le \beta \le \pi/2 \tag{2.10}$$

The solution of Eq. (2.9) can fall in the interval $[0, \pi/2]$ only in the case when $0 \leqslant \beta(z + 2n, r) \leqslant \alpha$ and $\varphi(z, r) = \alpha - \beta(z, r)$.

From Eqs (2.9) and (2.10) it follows that the inequality $\sin(\beta(z + 2n, r)) \leqslant \sin\alpha$ is equivalent to the inequality $z + 2n \leqslant r/N$, which imposes restrictions on the number n

$$n \le n(r, z) = [rN/2 - z/2] \tag{2.11}$$

(here $[x]$ is the integer part of the number n).

Let us put

$$q_n(r, z) = q(\varphi_n(r, z)), \quad \varphi_n(r, z) = \varphi(r, z+2n)$$

If condition (2.11) is satisfied, the function $f(r, z + 2n, t) \leqslant 0$ within the interval $[0, q_n(r, z)]$ by virtue of relations (2.7) and (2.5). Hence, in this interval the integrand in formula (2.5) vanishes. In the interval $[N_3, +\infty]$ the function $f(r, z + 2n, t) \geqslant 0$, the function $C(iq)$ takes imaginary values, whereas the functions $A(iq)$ and $B(iq)$ take real values. Therefore, in this interval the integrand in formula (2.5) also vanishes. From this it follows that the integration in formula (2.5) with respect to the variable q is actually carried out in the interval $[q_n(r, z), N_3]$.

Let us split the interval of integration into the intervals $[q_n(r, z), N_1]$ and $[N_1, N_3]$. The first of these intervals exists only if condition (2.11) is satisfied, and hence formula (2.5) can be rewritten in the form

$$\Phi_1(r, z, t) = \sum_{n=0}^{n(r,z)} \mathrm{Re} \int_{q_n(r,z)}^{N_1} \frac{\Gamma_n(q)\cos(qt)dq}{\sqrt{f(q, r, z+2n)}} + \sum_{n=0}^{\infty} \mathrm{Re} \int_{N_1}^{N_3} \frac{\Gamma_n(q)\cos(qt)dq}{\sqrt{f(q, r, z+2n)}} \tag{2.12}$$

3. ASYMPTOTIC FORMULAE AS $t \to \infty$

As is well known from general theory [2], the asymptotic form of integrals is determined by stationary and end points, as well as by points where the regularity of the integrand breaks down. We will confine ourselves to two terms of the asymptotic series, namely, the terms of order $t^{-1/2}$ and t^{-1}. The terms of order $t^{-1/2}$ are determined by the end point q_n. Applying the standard method [2] to formula (2.12), we obtain that, up to terms of order $t^{-3/2}$, the contribution from the point q_n to the asymptotic form of the function Φ_1 is equal to

$$\begin{aligned} \Phi_1(r, z, t) &= \sqrt{\frac{\pi}{t}} \sum_{n=0}^{n(r,z)} (-1)^{n+1} F_n(r, z)\cos\left(q_n(r, z)t + \frac{\pi}{4}\right) \\ F_n(r, z) &= \frac{q_n C(iq_n)A^n(iq_n)B^n(iq_n)}{\sqrt{\partial f(q_n(r, z), r, z+2n)/\partial q}} \end{aligned} \tag{3.1}$$

Introducing the notation

$$\psi^{\pm}(r, z) = 1 \pm N_1 \sin(\varphi_n(r, z))$$

$$\chi(r, z) = N_1(r^2 + (z+2n)^2 + c^2)\sin\varphi_n + c(z+2n)(1 + N_1^2 \sin^2\varphi_n)$$

$$B^{\pm}(r, z) = \sqrt{N_3^2 \psi^+ \psi^- + N_1^2} \pm \sqrt{1 - N_1^2}$$

after simple calculations we obtain

$$q_n C(iq_n) = \frac{N}{\psi^+}\cos\varphi_n, \quad A(iq_n) = \frac{\psi^-}{\psi^+}, \quad B(iq_n) = \frac{B^-(r, z)}{B^+(r, z)}$$

$$\frac{\partial f(q_n(r, z), r, z+2n)}{\partial q} = \frac{2\chi(r, z)\cos\varphi_n}{\sin\varphi_n \sqrt{\psi^+ \psi^-}}$$

$$F_n(r, z) = \frac{N}{2}\sqrt{\frac{\sin 2\varphi_n}{\chi}}(\psi^-)^{n+1/4}(\psi^+)^{-n-3/4}\left(\frac{B^-}{B^+}\right)^n$$

Substituting expressions (3.1) into formula (2.4), we arrive at the estimate

$$\int_0^T \frac{Q(\tau)}{\sqrt{t-\tau}}\cos\left(q_n(t-\tau) + \frac{\pi}{4}\right)d\tau = \frac{D_n(r, z)}{\sqrt{t}}\cos(q_n(r, z)t + p_n(r, z)) + O(t^{-3/2})$$
$$\int_0^T Q(\tau)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(q_n(r, z)\tau)d\tau = D_n(r, z)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(p_n(r, z)) \tag{3.2}$$

Substituting expressions (3.1) and (3.2) into formula (2.5), we obtain, to an accuracy of $O(t^{-3/2})$,

$$w_1 = -\frac{1}{4\pi^{5/2}\sqrt{t}}\sum_{n=0}^{n(r, z)}(-1)^{n+1}D_n F_n \cos\left(q_n(r, z)t + p_n(r, z) + \frac{\pi}{4}\right)$$

A similar estimate of the function w_2 is obtained by substituting $2 - z$ for z and $(B^-/B^+)F_n$ for F_n.

Up to terms $O(t^{-3/2})$, the contributions from the points N_1 and N_3 to the asymptotic form of the function Φ_1 can be found by integration by parts. In this case it turns out that these contributions are equal to zero.

REFERENCES

1. TER-KRIKOROV, A. M., Disturbances from a source in a three-layer atmosphere. *Prikl. Mat. Mekh.*, 2002, **66**, 1, 62–68.
2. FEDORYUK, M. V., *The Saddle-Point Method*. Nauka, Moscow, 1977.

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